

# Equating the achievable exponent region to the achievable entropy region by partitioning the source

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**Abstract**—In this paper we investigate the image size characterization problem. We show that any arbitrary source set may be decomposed into sets whose image size characterization is the same as its entropy characterization. We also show that the number of these sets required is small enough that one may consider that from a coding perspective the achievable entropy region and achievable exponent region are equal. This has an impact on many source networks and network problems whose solution heretofore could not have the image size characterization applied to them.

## I. INTRODUCTION

The image-size characterization problem was originally formulated by Körner and Marton in [1], and laid some of the groundwork for Marton's later paper on the broadcast channel [2] which established the best known inner bounds for the broadcast channel.

Consider a discrete memoryless multiple source (DMMS)  $(X^n, Y_1^n, \dots, Y_D^n)$  distributed according to the joint distribution  $P_{X, Y_1, \dots, Y_D}$ . One can define two different but similar problems [3, Ch. 15]. First the *entropy characterization problem* is the categorization of the region of all tuples  $(a, b_1, \dots, b_D)$  of  $D + 1$  non-negative numbers satisfying the condition that for every  $\delta > 0$  there exists a function  $f$  with domain  $\mathcal{X}^n$  such that

$$\left| \frac{1}{n} H(X^n | f(X^n)) - a \right| \leq \delta$$

$$\left| \frac{1}{n} H(Y_i^n | f(X^n)) - b_i \right| \leq \delta, \quad i = 1, 2, \dots, D,$$

whenever  $n$  is sufficiently large. The region spawned by this problem is referred to as  $\mathcal{F}$ . A related region is that of  $\mathcal{H}$ , which is the closure of  $\bigcup_{k=1}^{\infty} \mathcal{H}_k$ , where

$$\mathcal{H}_k \triangleq \left\{ \left( \frac{1}{k} H(X^k | U), \frac{1}{k} H(Y_1 | U), \dots, \frac{1}{k} H(Y_D | U) \right) : U \ominus X^k \ominus Y_1^k, \dots, Y_D^k \right\}.$$

Interestingly  $\mathcal{F} = \mathcal{H}$  [3, Problem 15.17] for every DMMS; thus characterization of both these regions may be considered as the entropy characterization problem. A single-letter solution to the entropy characterization problem is provided in [3, Ch. 15] for the special case of 3 component sources. Only

partial results [3, Problems 15.16–21] are available for the general case to date.

The second problem, which will be the primary focus of this paper, is the *image characterization problem*, where we categorize the tuples that for every  $\delta > 0$  and  $\eta \in (0, 1)$  there exists a set  $A \subseteq T_{[X]}^n$  such that

$$\left| \frac{1}{n} \log_2 |A| - a \right| \leq \delta$$

$$\left| \frac{1}{n} \log_2 g_{Y_i|X}^n(A, \eta) - b_i \right| \leq \delta, \quad i = 1, 2, \dots, D,$$

whenever  $n$  is large enough, where  $g_{Y_i|X}^n(A, \eta)$  is the minimum size of a set  $B_i \in \mathcal{Y}_i^n$  such that  $P_{Y_i|X}^n(B_i | x^n) \geq \eta$  for all  $x^n \in A$ . The corresponding region of tuples is denoted as  $\mathcal{G}$ . Note that in the image-size characterization problem, the conditional distributions  $P_{Y_i|X}, i = 1, 2, \dots, D$  describe  $D$  different discrete memoryless channels. A single-letter solution to the image-size characterization problem is provided in [3, Ch. 15] for the special case of 2 component channels. As Csiszár and Körner note in [3, pp. 339], “image size characterizations can be used to prove strong converse results for source networks and also to solve channel network problems. In this respect, it is important that the sets of achievable entropy resp. exponent triples have the same two dimensional projections.” It is however important to note that  $\mathcal{F} \neq \mathcal{G}$  in general.

Motivating this paper is that surprisingly, as shown in [3, Ch. 15], that for the case of 3 sources (or equivalently 2 channels), any triple  $(a, b, c) \in \mathcal{G}$  has the property that

$$(a, b, c) = \left( \max_{i=1,2} a_i, \max_{i=1,2} b_i, \max_{i=1,2} c_i \right),$$

for some triples  $(a_1, b_1, c_1)$  and  $(a_2, b_2, c_2) \in \mathcal{F}$ . The goal of this work is to show that not only may the image sizes be decomposed as such, but the sets themselves. Towards that goal, we show (cf. Theorem 2) there exists a partition of  $A = \bigcup A_k$  of no more than  $n\Gamma$  component sets, for some constant  $\Gamma$ , that the image size characterization problem restricting to each of these component sets is the same as the entropy characterization problem. To accomplish our goal we will use techniques associated with information spectrum [4]. We first present a new method of partitioning the destination sequence space based on information spectrum in section III. Then we

employ some consequences of such partitioning to show the main image size characterization results in section IV, and finish with a remark about the significance of these results in section V.

## II. NOTATION

To simplify writing, let  $[i : j]$  denote the set of integers starting at  $i$  and ending at  $j$ , inclusively. Consider a pair of discrete random variables  $X$  and  $Y$  over alphabets  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively. A set  $B \subseteq \mathcal{Y}^n$  is called an  $\eta$ -image of  $A \subseteq \mathcal{X}^n$  over the channel  $P_{Y|X}^n$  [3, Ch. 15] if  $P_{Y|X}^n(B|x^n) \geq \eta$  for every  $x^n \in A$ . On the other hand,  $B$  is called an  $\eta$ -quasi-image of  $A$  over the channel  $P_{Y|X}^n$  [3, Problem 15.13] if  $\Pr\{Y^n \in B | X^n \in A\} \geq \eta$ . The minimum size of  $\eta$ -images of  $A$  over  $P_{Y|X}^n$  will be denoted by  $g_{Y|X}^n(A, \eta)$ , while the minimum size of  $\eta$ -quasi-images of  $A$  over  $P_{Y|X}^n$  will be denoted by  $\bar{g}_{Y|X}^n(A, \eta)$ .

The notation for conditional entropy will be slightly abused throughout the paper. Within, when a quantity such as  $H(Y^n | X^n \in A')$  is expressed it will mean  $H(Y^n | E = 1)$ , where  $E$  is a random variable taking the value 1 if  $X^n \in A'$  and 0 if not. This is in contrast to the proper use in which  $H(Y^n | X^n \in A')$  would equal  $H(Y^n | E)$ .

## III. INFORMATION SPECTRUM PARTITION

Let  $p_n$  be a distribution on  $\mathcal{Y}^n$ , and  $i_n = -\frac{1}{n} \log_2 p_n$  be the corresponding information spectrum. For any  $\delta > 0$ , define  $K_\delta \triangleq \left\lceil \frac{\log_2 |\mathcal{Y}|}{\delta} \right\rceil$  and the  $\delta$ -information spectrum partition of  $\mathcal{Y}^n$  with respect to (w.r.t.)  $i_n$  be  $\{B_k\}_{k=0}^{K_\delta}$ , where

$$B_k \triangleq \begin{cases} \{y^n \in \mathcal{Y}^n : k\delta \leq i_n(y^n) < (k+1)\delta\} & \text{for } k \in [0 : K_\delta - 1] \\ \{y^n \in \mathcal{Y}^n : K_\delta \delta \leq i_n(y^n) < \infty\} & \text{for } k = K_\delta. \end{cases}$$

For convenience, we sometimes associate  $B_{-1} \triangleq \emptyset$  and the zero-probability set  $B_\infty \triangleq \{y^n : p_n(y^n) = 0\}$  with  $\{B_k\}$ . Clearly  $\mathcal{Y}^n = \left(\bigcup_{k=0}^{K_\delta} B_k\right) \cup B_\infty$ .

Fix any  $\delta \in (0, 1)$  and  $A \subseteq \mathcal{X}^n$  with the property that  $\Pr\{X^n \in A\} > 0$ . For the rest of this section, let  $P_A(y^n) \triangleq \Pr\{Y^n = y^n | X^n \in A\}$  and  $\{B_k\}$  be the  $\delta$ -information spectrum partition of  $\mathcal{Y}^n$  w.r.t.  $-\frac{1}{n} \log_2 P_A$ . We may derive the following consequences of information partitions.

**Lemma 1.** For every  $B_k$ ,  $k \in [0 : K_\delta]$ ,

$$\frac{1}{n} \log_2 |B_k| < (k+1)\delta.$$

In addition if  $P_A(B_k) > 2^{-n\delta}$  then

$$\left| \frac{1}{n} \log_2 |B_k| - k\delta \right| < \delta.$$

*Proof:* Trivially we have

$$\frac{1}{n} \log_2 |B_{K_\delta}| \leq \log_2 |\mathcal{Y}| < (K_\delta + 1)\delta.$$

For  $k \in [0 : K_\delta - 1]$ ,

$$1 \geq \sum_{y^n \in B_k} P_A(y^n) > |B_k| 2^{-n(k+1)\delta},$$

and therefore  $\frac{1}{n} \log_2 |B_k| < (k+1)\delta$ . Similarly suppose that  $P_A(B_k) = \sum_{y^n \in B_k} P_A(y^n) > 2^{-n\delta}$ . Then

$$2^{-n\delta} < \sum_{y^n \in B_k} P_A(y^n) \leq 2^{-nk\delta} |B_k|,$$

and therefore  $\frac{1}{n} \log_2 |B_k| > (k-1)\delta$ . Combining both results gives us  $\left| \frac{1}{n} \log_2 |B_k| - k\delta \right| < \delta$ . ■

**Lemma 2.** For any  $\eta \in (0, 1]$  and sufficiently large  $n$ , there exists a  $k' \in [0 : K_\delta]$  such that

$$\frac{1}{n} \log_2 \bar{g}_{Y|X}^n(A, \eta) \leq \frac{1}{n} \log_2 \left| \bigcup_{k=0}^{k'} B_k \right| \leq (k' + 2)\delta.$$

Furthermore if  $P_A(B_{k'-1}) > 2^{-n\delta}$ , then

$$\frac{1}{n} \log_2 \bar{g}_{Y|X}^n(A, \eta) \geq \frac{1}{n} \log_2 |B_{k'-1}| \geq (k' - 2)\delta.$$

*Proof:* Let  $\eta_{-1} \triangleq 0$  and  $\eta_{k'} \triangleq P_A\left(\bigcup_{k=0}^{k'} B_k\right)$  for  $k' \in [0 : K_n]$ . Note then that  $0 = \eta_{-1} \leq \eta_0 \leq \eta_1 \leq \dots \leq \eta_{K_\delta} = 1$ . In addition  $\eta_{k-1} = \eta_k$  implies  $B_k = \emptyset$ . Write  $B' = \bigcup_{k=0}^{k'} B_k$  to simplify notation below. Clearly  $B'$  is an  $\eta_{k'}$ -quasi-image of  $A$ . We claim that  $B'$  is in fact the unique  $\eta_{k'}$ -quasi-image of  $A$  that achieves the minimum size  $\bar{g}_{Y|X}^n(A, \eta_{k'})$ . To show the claim, consider a set  $\hat{B} \subseteq \mathcal{Y}^n$  such that  $\hat{B} \neq B'$  and  $|\hat{B}| \leq |B'|$ . For  $k' = K_\delta$ ,  $\hat{B}$  clearly cannot be the  $\eta_{k'}$ -quasi-image of  $A$ . On the other hand, for  $k' \in [0 : K_\delta - 1]$ ,

$$\begin{aligned} P_A(\hat{B}) &= P_A(B') - P_A(B' \setminus \hat{B}) + P_A(\hat{B} \setminus B') \\ &= \eta_{k'} - \sum_{k=0}^{k'} \sum_{y^n \in B_k \setminus \hat{B}} P_A(y^n) + \sum_{k \geq k'+1} \sum_{y^n \in B_k \cap \hat{B}} P_A(y^n) \\ &< \eta_{k'} - \left(|B'| - |B' \cap \hat{B}|\right) 2^{-n(k'+1)\delta} \\ &\quad + \left(|\hat{B}| - |B' \cap \hat{B}|\right) 2^{-n(k'+1)\delta} \leq \eta_{k'}. \end{aligned}$$

Thus  $\hat{B}$  cannot be an  $\eta_{k'}$ -quasi-image of  $A$ .

Next it is clear that for any  $\eta \in (0, 1]$  there exists a  $k'$  such that  $\eta_{k'-1} < \eta \leq \eta_{k'}$  which gives us that

$$\frac{1}{n} \log_2 \left| \bigcup_{k=0}^{k'-1} B_k \right| \leq \frac{1}{n} \log_2 \bar{g}_{Y|X}^n(A, \eta) \leq \frac{1}{n} \log_2 \left| \bigcup_{k=0}^{k'} B_k \right|. \quad (1)$$

Lemma 1 and the upper bound in (1) give us that

$$\frac{1}{n} \log_2 \bar{g}_{Y|X}^n(A, \eta) \leq \frac{1}{n} \log_2 \sum_{k=0}^{k'} 2^{n(k+1)\delta} \leq (k' + 2)\delta$$

when  $n$  is sufficiently large. Furthermore if  $P_A(B_{k'-1}) > 2^{-n\delta}$ , then combining the lower bound of (1) and Lemma 1 again we have

$$\frac{1}{n} \log_2 \bar{g}_{Y|X}^n(A, \eta) \geq \frac{1}{n} \log_2 |B_{k'-1}| \geq (k' - 2)\delta.$$

**Lemma 3.** Fix any  $\alpha' \in (0, 1)$ . For any  $\alpha_n \in (0, \alpha']$  with  $\frac{-\log_2 \alpha_n}{n} \rightarrow 0$ , there exist  $\beta_n \rightarrow 1$  and  $\tau_n \rightarrow 0$  such that

$$0 \leq \frac{1}{n} \log_2 g_{Y|X}^n(A', \beta_n) - \frac{1}{n} \log_2 g_{Y|X}^n(A', \alpha_n) \leq \tau_n$$

for every  $A' \subseteq \mathcal{X}^n$ , whenever  $n$  is sufficiently large. Furthermore the same  $\beta_n$  and  $\tau_n$  can be used uniformly for all  $\alpha_n \geq \frac{1}{n^2}$ .

*Proof:* This is a slightly strengthened version of [3, Lemma 6.6], whose proof (cf. also [3, Ch. 5]) directly applies to the current lemma. ■

**Lemma 4.** Let  $X^n$  be conditionally uniformly distributed over  $A$ . Then for any  $\alpha_n \in (0, 1]$  with  $\frac{-\log_2 \alpha_n}{n} \rightarrow 0$ , there exist  $A' \subseteq A$ ,  $\tau_n \rightarrow 0$ , and  $\beta_n \rightarrow 1$  such that  $\frac{|A'|}{|A|} \geq (1 - \frac{1}{n}) \alpha_n$  and

$$\frac{1}{n} \log_2 g_{Y|X}^n(A', \beta_n) \leq \frac{1}{n} \log_2 \bar{g}_{Y|X}^n(A, \alpha_n) + \tau_n,$$

whenever  $n$  is sufficiently large. Neither  $\tau_n$  nor  $\beta_n$  depends on  $A$ . Furthermore neither depends on  $\alpha_n$  if  $\alpha_n \geq \frac{1}{n}$ .

*Proof:* Let  $B \subseteq \mathcal{Y}^n$  be an  $\alpha_n$ -quasi-image of  $A$  that achieves  $\bar{g}_{Y|X}^n(A, \alpha_n)$ . Define

$$A' \triangleq \left\{ x^n \in A : P_{Y|X}^n(B|x^n) \geq \frac{\alpha_n}{n} \right\}.$$

Clearly  $B$  is an  $\frac{\alpha_n}{n}$ -image of  $A'$ . Hence

$$\begin{aligned} \frac{1}{n} \log_2 \bar{g}_{Y|X}^n(A, \alpha_n) &= \frac{1}{n} \log_2 |B| \\ &\geq \frac{1}{n} \log_2 g_{Y|X}^n(A', \frac{\alpha_n}{n}) \\ &\geq \frac{1}{n} \log_2 g_{Y|X}^n(A', \beta_n) - \tau_n \end{aligned}$$

by Lemma 3 since  $\frac{\log_2 n - \log_2 \alpha_n}{n} \rightarrow 0$ . Note that the same  $\beta_n$  and  $\tau_n$  can be used uniformly for all  $\alpha_n \geq \frac{1}{n}$ .

Further as  $B$  is an  $\alpha_n$ -quasi-image of  $A$ , we have

$$\begin{aligned} \alpha_n &\leq P_A(B) \\ &= \frac{1}{|A|} \sum_{x^n \in A} P_{Y|X}^n(B|x^n) \\ &= \frac{1}{|A|} \sum_{x^n \in A'} P_{Y|X}^n(B|x^n) + \frac{1}{|A|} \sum_{x^n \in A \setminus A'} P_{Y|X}^n(B|x^n) \\ &\leq \frac{|A'|}{|A|} + \left(1 - \frac{|A'|}{|A|}\right) \frac{\alpha_n}{n} \end{aligned}$$

which implies  $\frac{|A'|}{|A|} \geq (1 - \frac{1}{n}) \alpha_n$ . ■

**Lemma 5.** Suppose that  $X^n$  is conditionally uniformly distributed on  $A$ . Then there exist  $A^* \subseteq A$ ,  $\varepsilon_n \rightarrow 0$ , and  $\beta_n \rightarrow 1$  satisfying  $\frac{|A^*|}{|A|} \geq \frac{1}{2(K_\delta + 1)}$  and

$$\frac{1}{n} H(Y^n | X^n \in A^*) \geq \frac{1}{n} \log_2 g_{Y|X}^n(A^*, \beta_n) - 7.19\delta - \varepsilon_n,$$

whenever  $n$  is sufficiently large. Neither  $\varepsilon_n$  nor  $\beta_n$  depends on  $A$ .

*Proof:* Define  $\eta_k \triangleq P_A\left(\bigcup_{l=0}^k B_l\right)$  for  $k \in [0 : K_n]$  as in the proof of Lemma 2. Because the total number of sets in  $\{B_k\}$  is  $K_\delta + 1$ , we know that there exists at least one  $k' \in [0, K_\delta]$  such that  $P_A(B_{k'}) \geq \frac{1}{K_\delta + 1}$ . Apply Lemma 4 by choosing  $\alpha_n = \eta_{k'} \geq \frac{1}{2(K_\delta + 1)}$  to obtain  $\tau_n \rightarrow 0$ ,  $\beta_n \rightarrow 1$ , and  $A' \subseteq A$  that satisfy

$$\frac{|A'|}{|A|} \geq \left(1 - \frac{1}{n}\right) \eta_{k'} \geq \frac{1}{2(K_\delta + 1)}, \quad (2)$$

$$\frac{1}{n} \log_2 g_{Y|X}^n(A', \beta_n) \leq \frac{1}{n} \log_2 \bar{g}_{Y|X}^n(A, \eta_{k'}) + \tau_n, \quad (3)$$

whenever  $n$  is sufficiently large. Note that the  $\beta_n$  and  $\tau_n$  above are the ones that work uniformly for all  $\alpha_n \geq \frac{1}{n}$  in Lemma 4.

First consider the case of  $k' \leq c_n \triangleq 4.19 + \frac{\tau_n}{\delta}$ . From (3),

$$\begin{aligned} \frac{1}{n} \log_2 g_{Y|X}^n(A', \beta_n) &\leq \frac{1}{n} \log_2 \bar{g}_{Y|X}^n(A, \eta_{k'}) + \tau_n \\ &\stackrel{(a)}{=} \frac{1}{n} \log_2 \left| \bigcup_{k=0}^{k'} B_k \right| + \tau_n \leq (k' + 2)\delta + \tau_n \\ &\leq 6.19\delta + 2\tau_n. \end{aligned} \quad (4)$$

where (a) is due to the fact that  $\bigcup_{k=0}^{k'} B_k$  is the  $\eta_{k'}$ -quasi-image of  $A$  that achieves  $\bar{g}_{Y|X}^n(A, \eta_{k'})$  as shown in the proof of Lemma 2. Since  $H(Y^n | X^n \in A') \geq 0$ , the conclusions of the lemma are clearly satisfied.

It remains to consider the case of  $k' > c_n$ . To that end, let  $k'' \triangleq \lfloor k' - c_n \rfloor$ , and define set  $\tilde{B} = \bigcup_{k=0}^{k''} B_k$ . First assume that  $P_A(\tilde{B}) = \eta_{k''} > \frac{1}{n}$ . Apply Lemma 4 again with  $\alpha_n = \eta_{k''} > \frac{1}{n}$  to obtain  $A'' \subseteq A$  that satisfies

$$\begin{aligned} \frac{1}{n} \log_2 g_{Y|X}^n(A'', \beta_n) &\leq \frac{1}{n} \log_2 \bar{g}_{Y|X}^n(A, \frac{1}{n}) + \tau_n \\ &\leq \frac{1}{n} \log_2 \bar{g}_{Y|X}^n(A, \eta_{k''}) + \tau_n \stackrel{(a)}{\leq} (k'' + 2)\delta + \tau_n \\ &\leq (k' - 2.19)\delta \stackrel{(b)}{\leq} \frac{1}{n} \log_2 |B_{k'}| - 1.19\delta \end{aligned} \quad (5)$$

where (a) and (b) are due to Lemmas 2 and 1, respectively.

Let  $\hat{B}$  be the  $\beta_n$ -image of  $A''$  that achieves  $g_{Y|X}^n(A'', \beta_n)$ . By definition, every  $y^n \in B_{k'}$  has the property that  $2^{-n(k'+1)\delta} < P_A(y^n) \leq 2^{-nk'\delta}$ . This implies

$$\begin{aligned} P_A(B_{k'} \setminus \hat{B}) &= P_A(B_{k'}) - P_A(B_{k'} \cap \hat{B}) \\ &\geq \frac{1}{K_\delta + 1} - 2^{-nk'\delta} |B_{k'} \cap \hat{B}| \\ &\geq \frac{1}{K_\delta + 1} - 2^{-nk'\delta} g_{Y|X}^n(A'', \beta_n) \\ &\geq \frac{1}{K_\delta + 1} - 2^{n\delta} \frac{g_{Y|X}^n(A'', \beta_n)}{|B_{k'}|} \\ &\geq \frac{1}{K_\delta + 1} - 2^{-0.19n\delta} \end{aligned}$$

where the second last and last inequalities are due to Lemma 1 and (5), respectively. Continuing on,

$$\frac{1}{K_\delta + 1} - 2^{-0.19n\delta} \leq P_A(B_{k'} \setminus \hat{B})$$

$$\begin{aligned}
&= \frac{1}{|A|} \sum_{x^n \in A} P_{Y|X}^n(B_{k'} \setminus \hat{B}|x^n) \\
&\stackrel{(a)}{=} \frac{1}{|A|} \sum_{x^n \in A''} P_{Y|X}^n(B_{k'} \setminus \hat{B}|x^n) \\
&\quad + \frac{1}{|A|} \sum_{x^n \in A' \setminus A''} P_{Y|X}^n(B_{k'} \setminus \hat{B}|x^n) \\
&\quad + \frac{1}{|A|} \sum_{x^n \in A \setminus (A' \cup A'')} P_{Y|X}^n(B_{k'} \setminus \hat{B}|x^n) \\
&\stackrel{(b)}{\leq} (1 - \beta_n) + \frac{|A' \setminus A''|}{|A|} + \frac{\eta_{k'}}{n}
\end{aligned}$$

where each term in (b) bounds the corresponding term in (a). In particular, the first bound in (b) is due to the fact that each  $x^n \in A''$  satisfies  $P_{Y|X}^n(\hat{B}^c|x^n) < 1 - \beta_n$ . On the other hand, the third bound in (b) results from the fact that  $A'$  contains all  $x^n \in A$  that  $P_{Y|X}^n\left(\bigcup_{k=0}^{k'} B_k|x^n\right) \geq \frac{\eta_{k'}}{n}$  as defined in the proof of Lemma 4 because  $\bigcup_{k=0}^{k'} B_k$  is the unique minimum-cardinality  $\eta_{k'}$ -quasi-image of  $A$  (cf. the proof of Lemma 2). As a result, we have

$$\begin{aligned}
\frac{|A' \setminus A''|}{|A|} &\geq \frac{1}{K_\delta + 1} - 2^{-0.19n\delta} - (1 - \beta_n) - \frac{1}{n} \\
&\geq \frac{1}{2(K_\delta + 1)}
\end{aligned} \tag{6}$$

for all sufficiently large  $n$ . Now since  $X^n$  is conditionally uniform in  $A$ , we have

$$\begin{aligned}
P_{A' \setminus A''}(y^n) &\triangleq \Pr(Y^n = y^n | X^n \in A' \setminus A'') \\
&= \frac{1}{|A' \setminus A''|} \sum_{x^n \in A' \setminus A''} P_{Y|X}^n(y^n|x^n) \\
&\leq \frac{2(K_\delta + 1)}{|A|} \sum_{x^n \in A} P_{Y|X}^n(y^n|x^n) \leq 2(K_\delta + 1)P_A(y^n). \tag{7}
\end{aligned}$$

Hence using (7) we get

$$\begin{aligned}
&\frac{1}{n}H(Y^n|X^n \in A' \setminus A'') \\
&\geq -\frac{1}{n} \sum_{y^n \notin \tilde{B}} P_{A' \setminus A''}(y^n) \log_2 P_{A' \setminus A''}(y^n) \\
&\geq -\frac{\log_2 2(K_\delta + 1)}{n} - \frac{1}{n} \sum_{k=k''}^{K_\delta} \sum_{y^n \in B_k} P_{A' \setminus A''}(y^n) \log_2 P_A(y^n) \\
&\geq -\frac{\log_2 2(K_\delta + 1)}{n} + \sum_{k=k''}^{K_\delta} P_{A' \setminus A''}(B_k) \cdot k\delta \\
&\geq -\frac{\log_2 2(K_\delta + 1)}{n} + (k' - c_n - 1)\delta P_{A' \setminus A''}(\tilde{B}^c) \\
&\stackrel{(a)}{\geq} -\frac{\log_2 2(K_\delta + 1)}{n} + \left( \frac{1}{n} \log_2 g_{Y|X}^n(A', \beta_n) - 7.19\delta \right. \\
&\quad \left. - 2\tau_n \right) \cdot P_{A' \setminus A''}(\tilde{B}^c) \\
&\stackrel{(b)}{\geq} -\frac{\log_2 2(K_\delta + 1)}{n} + \left( \frac{1}{n} \log_2 g_{Y|X}^n(A' \setminus A'', \beta_n) \right.
\end{aligned}$$

$$\begin{aligned}
&\quad \left. - 7.19\delta - 2\tau_n \right) \cdot \left( 1 - \frac{\eta_{k''}}{n} \right) \\
&\geq \frac{1}{n} \log_2 g_{Y|X}^n(A' \setminus A'', \beta_n) - \frac{\log_2 [2(K_\delta + 1) \log_2 |\mathcal{Y}|]}{n} \\
&\quad - 7.19\delta - 2\tau_n
\end{aligned} \tag{8}$$

where (a) is due to (4) and (b) is due to the fact that  $A''$  contains all  $x^n \in A$  that  $P_{Y|X}^n(\tilde{B}|x^n) \geq \frac{\eta_{k''}}{n}$ . Clearly then the conclusions of the lemma result from (6) and (8).

Finally if  $P_A(\tilde{B}) \leq \frac{1}{n}$ , then following the same development from (6) to (8) based on (2), we get  $P_{A'}(y^n) \leq 2(K_\delta + 1)P_A(y^n)$  and

$$\begin{aligned}
&\frac{1}{n}H(Y^n|X^n \in A') \\
&\geq -\frac{\log_2 2(K_\delta + 1)}{n} + \left( \frac{1}{n} \log_2 g_{Y|X}^n(A', \beta_n) \right. \\
&\quad \left. - 7.19\delta - 2\tau_n \right) \cdot \left( 1 - 2(K_\delta + 1)P_A(\tilde{B}) \right) \\
&\geq \frac{1}{n} \log_2 g_{Y|X}^n(A', \beta_n) - \frac{\log_2 2(K_\delta + 1)}{n} \\
&\quad - \frac{2(K_\delta + 1) \log_2 |\mathcal{Y}|}{n} - 7.19\delta - 2\tau_n.
\end{aligned}$$

This, together with (2), again gives the lemma.  $\blacksquare$

#### IV. IMAGE SIZE CHARACTERIZATION

**Theorem 1.** Fix any  $\eta \in (0, 1)$  and  $\epsilon > 0$ . Let  $X^n$  be uniformly distributed over any  $A \subseteq \mathcal{X}^n$ . For  $i \in [1 : D]$ , suppose that  $Y_i^n$  is conditionally distributed according to the channel  $P_{Y_i|X}^n$  given  $X^n$ . Then there exists  $A' \subseteq A$  satisfying

- 1)  $0 \leq \frac{1}{n} \log_2 |A| - \frac{1}{n} \log_2 |A'| \leq \epsilon$ ,
- 2)  $\frac{1}{n} H(X^n|X^n \in A') = \frac{1}{n} \log_2 |A'|$ , and
- 3)  $\left| \frac{1}{n} H(Y_i^n|X^n \in A') - \frac{1}{n} \log_2 g_{Y_i|X}^n(A', \eta) \right| \leq \epsilon$  for  $i \in [1 : D]$ ,

whenever  $n$  is sufficiently large.

*Proof:* We give the proof for the cases of  $D = 1$  and 2 below. The proof naturally extends for  $D > 2$ .

Apply Lemma 5 based on the  $\delta_1$ -information spectrum partition of  $\mathcal{Y}_1^n$  to obtain  $A_1 \subseteq A$  and  $\epsilon_n \rightarrow 0$  such that

$$\begin{aligned}
&\frac{1}{n} \log_2 |A| - \frac{1}{n} \log_2 |A_1| \leq \frac{1}{n} \log_2 2(K_{\delta_1} + 1) \\
&\frac{1}{n} \log_2 g_{Y_1|X}^n(A_1, \eta) \leq \frac{1}{n} H(Y_1^n|X^n \in A_1) + \epsilon_n + 7.19\delta_1,
\end{aligned} \tag{9}$$

for all sufficiently large  $n$ . On the other hand, for all sufficiently large  $n$ , by [3, Lemma 15.2]

$$\frac{1}{n} H(Y_1^n|X^n \in A_1) \leq \frac{1}{n} \log_2 g_{Y_1|X}^n(A_1, \eta) + \epsilon. \tag{11}$$

Note that (9), (10), and (11) together with a small enough  $\delta_1$  establish the theorem for the case of  $D = 1$ .

Next apply Lemma 5 based on the  $\delta_2$ -information spectrum partition of  $\mathcal{Y}_2^n$  and [3, Lemma 15.2] again to obtain  $A_2 \subseteq A_1$  such that

$$\frac{1}{n} \log_2 |A_1| - \frac{1}{n} \log_2 |A_2| \leq -\frac{1}{n} \log_2 2(K_{\delta_2} + 1), \tag{12}$$

$$\frac{1}{n} \log_2 g_{Y_2|X}^n(A_2, \eta) \leq \frac{1}{n} H(Y_2^n | X^n \in A_2) + \varepsilon_n + 7.19\delta_2, \quad (13)$$

$$\frac{1}{n} H(Y_2^n | X^n \in A_2) \leq \frac{1}{n} \log_2 g_{Y_2|X}^n(A_2, \eta) + \epsilon. \quad (14)$$

whenever  $n$  is sufficiently large. Furthermore, applying [3, Lemma 15.2] on  $A_2$  and  $A_1 \setminus A_2$  over the first channel gives us, respectively,

$$\frac{1}{n} H(Y_1^n | X^n \in A_2) \leq \frac{1}{n} \log_2 g_{Y_1|X}^n(A_2, \eta) + \epsilon, \quad (15)$$

$$\begin{aligned} & \frac{1}{n} H(Y_1^n | X^n \in A_1 \setminus A_2) \\ & \leq \frac{1}{n} \log_2 g_{Y_1|X}^n(A_1 \setminus A_2, \eta) + \frac{\epsilon}{4(K_{\delta_2} + 1)} \\ & \leq \frac{1}{n} H(Y_1^n | X^n \in A_1) + \varepsilon_n + 7.19\delta_1 + \frac{\epsilon}{4(K_{\delta_2} + 1)} \end{aligned} \quad (16)$$

where the last inequality is due to (10).

Now let  $S$  be in the indicator random variable of the event that  $X^n \in A_2$ . We have

$$\begin{aligned} H(Y_1^n | X^n \in A_1) &= I(S; Y_1^n | X^n \in A_1) + H(Y_1^n | S, X^n \in A_1) \\ &\leq 1 + H(Y_1^n | X^n \in A_2) \Pr\{X^n \in A_2 | X^n \in A_1\} \\ &\quad + H(Y_1^n | X^n \in A_1 \setminus A_2) \Pr\{X^n \in A_1 \setminus A_2 | X^n \in A_1\} \\ &\leq 1 + H(Y_1^n | X^n \in A_2) \cdot \frac{|A_2|}{|A_1|} + \left[ H(Y_1^n | X^n \in A_1) \right. \\ &\quad \left. + n\varepsilon_n + 7.19n\delta_1 + \frac{n\epsilon}{4(K_{\delta_2} + 1)} \right] \cdot \left[ 1 - \frac{|A_2|}{|A_1|} \right] \end{aligned} \quad (17)$$

where the last inequality is due to (16). Because of (12), we can rearrange (17) to get

$$\begin{aligned} & \frac{1}{n} H(Y_1^n | X^n \in A_2) \\ & \geq \frac{1}{n} H(Y_1^n | X^n \in A_1) + 7.19\delta_1 + \varepsilon_n \\ & \quad - \left( \varepsilon_n + 7.19\delta_1 + \frac{\epsilon}{4(K_{\delta_2} + 1)} + \frac{1}{n} \right) \frac{|A_1|}{|A_2|} \\ & \geq \frac{1}{n} \log_2 g_{Y_1|X}^n(A_2, \eta) - \frac{\epsilon}{2} \\ & \quad - 2(K_{\delta_2} + 1) \left( \varepsilon_n + 7.19\delta_1 + \frac{1}{n} \right) \end{aligned} \quad (18)$$

where the last inequality is due to (10) and (12). Finally putting (9), (12), (13), (14), (15), and (18) together with small enough  $\delta_1$ ,  $\delta_2$ , and  $\frac{\delta}{\delta_2}$ , we get the theorem for the case of  $D = 2$ . ■

**Theorem 2.** Fix any  $\eta \in (0, 1)$  and  $\epsilon > 0$ . Let  $X^n$  be uniformly distributed over any  $A \subseteq \mathcal{X}^n$ . For  $i \in [1 : D]$ , suppose that  $Y_i^n$  is conditionally distributed according to the channel  $P_{Y_i|X}^n$  given  $X^n$ . Then there exist a constant  $\Gamma > 0$  and a partition of  $A = \bigcup_{k=1}^m A_k$  with  $m \leq n\Gamma$  that satisfies

- 1)  $\frac{1}{n} H(X^n | X^n \in A_k) = \frac{1}{n} \log_2 |A_k|$  and
- 2)  $\left| \frac{1}{n} H(Y_i^n | X^n \in A_k) - \frac{1}{n} \log_2 g_{Y_i|X}^n(A_k, \eta) \right| \leq \epsilon$  for  $i \in [1 : D]$ ,

for all  $k \in [1 : m]$ , whenever  $n$  is sufficiently large.

*Proof:* Using Theorem 1 on  $A$ , we immediately obtain  $A_1 \subseteq A$  that satisfies 1) and 2). In addition,  $\Pr\{X^n \in A \setminus A_1\} \leq \delta$  for some  $\delta \in (0, 1)$ . Next apply Theorem 1 again on  $A \setminus A_1$ , we get  $A_2 \subseteq A \setminus A_1$  satisfying 1), 2), and  $\Pr\{X^n \in A \setminus (A_1 \cup A_2) | X^n \in A \setminus A_1\} \leq \delta$ . Repeat this process  $m - 2$  more times to get  $A_k \subseteq A \setminus \bigcup_{j=1}^{k-1} A_j$  satisfying 1), 2), and

$$\Pr\left\{X^n \in A \setminus \bigcup_{j=1}^k A_j \mid X^n \in A \setminus \bigcup_{j=1}^{k-1} A_j\right\} \leq \delta$$

for  $k \in [3 : m]$ . Write  $\tilde{A} \triangleq \bigcup_{j=1}^m A_j$ . Then combining the conditional probability bounds above, we have  $\Pr\{X^n \in \tilde{A}\} \leq \delta^m$ . Since  $X^n$  distributed uniformly in  $A$ ,  $\Pr\{X^n \in \tilde{A}\} \geq 2^{-n \log_2 |\mathcal{X}|}$ . Thus  $\tilde{A}$  must be empty when  $m > \frac{n}{-\log_{|\mathcal{X}|} \delta}$ . ■

## V. CONCLUDING REMARK

Consider a coding application in which the set  $A \subseteq \mathcal{X}^n$  represents the codebook. Theorem 2 tells us that  $A$  can be broken down into  $\bigcup_{k=1}^m A_k$  of at most  $n\Gamma$  sets. Let  $E = k$  if  $X^n \in A_k$  for  $k \in [1 : m]$ , and hence  $H(E) \leq \log_2 n + \log_2 \Gamma$ . For any message  $M$  of rate  $R$  carried by the codebook  $A$  to be received at receiver  $i$ , by Fano's Inequality we have

$$\begin{aligned} R &\leq \frac{1}{n} I(M; Y_i^n) + \frac{1}{n} + P_e R \\ &\leq I(M; Y_i^n | E) + \frac{1 + \log_2 n}{n} + P_e R \end{aligned}$$

where  $P_e$  is the error probability of decoding  $M$  based on observing  $Y_i^n$ . Thus it suffices to restrict to those codewords with each  $A_k$ . Within each  $A_k$  the image-size characterization, which is important to further bounding  $I(M; Y_i^n | E = k)$ , is the same as the achievable entropy characterization. Conversely, suppose that for each  $k$ , one has a coding scheme to send  $M$ , restricted to  $A_k$ , through the  $i$ th channel that achieves the rate  $\frac{1}{n} I(M; Y_i^n | E = k)$ . Then one can derive a scheme to first send  $E$  and then send  $M$  to achieve rate  $\frac{1}{n} I(M; Y_i^n | E)$  because the number of bits needed to communicate  $E$  is negligible compared to that required to send  $M$ . Once again the achievability of  $\frac{1}{n} I(M; Y_i^n | E = k)$  depends on the image-size characterization, which on  $A_k$  is the same as the achievable entropy characterization.

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